MATH 54 – MIDTERM 2 – SOLUTIONS

PEYAM RYAN TABRIZIAN

1. *(10 points, 2 points each)*

Label the following statements as **T** or **F**.

NOTE: In this question, you do **NOT** have to show your work! Don't spend *too* much time on each question!

(a) $\boxed{\textbf{FALSE}}$ If A is a $m \times n$ matrix, then $dim(Nul(A)) + Rank(A) = m$

(it's $dim(Nul(A)) + Rank(A) = n$, not m)

(b) $\boxed{\textbf{FALSE}}$ The change-of-coordinates matrix P from $\mathcal{B}$ to $\mathcal{C}$ has the property that $[\mathbf{x}]_{\mathcal{B}} = P\,[\mathbf{x}]_{\mathcal{C}}$ ($P = \mathcal{C} \overset{P}{\leftarrow} \mathcal{B}$, so $[\mathbf{x}]_{\mathcal{C}} = \mathcal{C} \overset{P}{\leftarrow} \mathcal{B}$ $[\mathbf{x}]_{\mathcal{B}}$)

(c) $\boxed{\textbf{FALSE}}$ If $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$, then $Nul(T) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

$$Nul(T) = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \text{ s.t. } \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \text{ s.t.} x = 0\right\} = \left\{\begin{bmatrix} 0 \\ y \end{bmatrix}\right\} = Span\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

(d) $\boxed{\textbf{TRUE}}$ The set of polynomials $\mathbf{p}$ in P_2 such that $\mathbf{p}(3) = 0$ is a subspace of P_2

(You can easily check that the 0-polynomial is in it, that it is closed under addition and scalar multiplication)

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(e) $\boxed{\textbf{FALSE}}$ $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$ (it's not even a *subset* of $\mathbb{R}^3$!!!)

2. *(20 points, 5 points each)* Label the following statements as **TRUE** or **FALSE**.

(a) $\boxed{\textbf{TRUE}}$ The set of matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is a subspace of $M_{2\times 2}$.
If you denote that set by V, then you get:

$$V = Span \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

And since the span of anything is a vector space, V is a vector space, and hence a subspace of $M_{2\times 2}$.

Alternatively you could have shown in the usual way that the O matrix is in it, and that it is closed under addition and scalar multiplication.

(b) $\boxed{\textbf{TRUE}}$ The matrix of the linear transformation T which reflects points about the x-axis and then about the y-axis is the same as the matrix of the linear transformation S which rotates points about the origin by 180 degrees counterclockwise.

Calculate $T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$
Hence the matrix of T is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Calculate $S\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $S\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$
Hence the matrix of S is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

And notice the two matrices are the same!

(c) **TRUE** The following set is a basis for P_2: $\{1, 1+t, 1+t+t^2\}$.

Linear independence: Suppose $a(1)+b(1+t)+c(1+t+t^2) = 0$, then $(a+b+c)+(b+c)t+ct^2 = 0$, hence $c = 0$, hence $b = 0$, hence $a = 0$, hence $a = b = c = 0$, and the polynomials are linearly independent.

Span: Since P_2 is 3-dimensional, and the set contains 3 elements, hence the set also spans P_2

Therefore the set is a basis for P_2.

Note: There were many, many, many other ways to show why this is true! One way is to consider the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and notice its determinant is $1 \neq 0$, hence it is invertible, hence its columns are linearly independent and span $\mathbb{R}^3$. Or you could use the Wronskian (if you want to make me happy :)).

(d) **FALSE** If V is a set that contains the $\mathbf{0}$-vector, and such that whenever $\mathbf{u}$ and $\mathbf{v}$ are in V, then $\mathbf{u}+\mathbf{v}$ is in V, then V is a vector space!

Consider the set $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, y \geq 0 \right\}$ in $\mathbb{R}^2$. (i.e. the upper-half-plane)

$\mathbf{0}$-vector: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is in it!

Closed under addition: Suppose $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ are in V, then $y \geq 0$ and $y' \geq 0$. Then $\mathbf{u}+\mathbf{v} = \begin{bmatrix} x+x' \\ y+y' \end{bmatrix}$. But since $y+y' \geq 0$, we get $\mathbf{u}+\mathbf{v}$ is in V

Not closed under scalar multiplication: For example, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in V, but $(-2)\mathbf{u} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ is not in V.

Note: I'm not going to be *too* harsh if you forgot to show that it's closed under addition, but I'll be very harsh if you didn't explicitly show to me that it's closed under scalar multiplication!!!

3. *(10 points)* For the following matrix A, find a basis for $Nul(A)$, $Row(A)$, $Col(A)$, and find $Rank(A)$:

$$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$Nul(A)$ Since the right-hand-side is not in reduced row-echelon form, let's further row-reduce it:

$$\begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 4 & 5 & -6 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 & 9 & 2 \\ 0 & 1 & -1 & 0 & 7 & 3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(I first subtracted the second row from the first, and then subtracted 3 times the third row from the second and 4 times the third row from the first)

Now if $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ t \\ s \\ r \end{bmatrix}$, then we get:

$$\begin{cases} x - 2z + 9s + 2r = 0 \\ y - z + 7s + 3r = 0 \\ t - s - 2r = 0 \end{cases}$$

That is:

$$\begin{cases} x = 2z - 9s - 2r \\ y = z - 7s - 3r \\ t = s + 2r \end{cases}$$

Hence we get:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ t \\ s \\ r \end{bmatrix} = \begin{bmatrix} 2z - 9s - 2r \\ z - 7s - 3r \\ z \\ s + 2r \\ s \\ r \end{bmatrix} = z \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

And therefore:

$$Nul(A) = Span \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\underline{Row(A)}$ Notice that there are pivots in the first, second, and third row, hence:

$$Row(A) = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ -3 \\ 7 \\ 9 \\ -9 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 3 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ -2 \end{bmatrix} \right\}$$

$\underline{Col(A)}$ Notice that there are pivots in the first, second, and fourth columns, hence:

$$Col(A) = Span\left\{\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\\-3\\-2\end{bmatrix}, \begin{bmatrix}7\\10\\1\\-5\\0\end{bmatrix}\right\}$$

(Notice that you had to go back to the matrix A to find a basis for $Col(A)$)

$\underline{Rank(A)}$ There are 3 pivots, hence $Rank(A) = 3$.

4. *(10 points)* Let $\mathcal{B} = \{7 - 2t, 2 - t\}$, and $\mathcal{C} = \{4 + t, 5 + 2t\}$ be bases for P_1.

Calculate $[\mathbf{x}]_{\mathcal{C}}$ given $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}1\\4\end{bmatrix}$.

Hint: First calculate a change-of-coordinates matrix!

First of all, identifying polynomials with their number codes, we get $\mathcal{B} = \left\{\begin{bmatrix}7\\-2\end{bmatrix}, \begin{bmatrix}2\\-1\end{bmatrix}\right\}$ and $\mathcal{C} = \left\{\begin{bmatrix}4\\1\end{bmatrix}, \begin{bmatrix}5\\2\end{bmatrix}\right\}$.

$$[\mathcal{C} \mid \mathcal{B}] = \begin{bmatrix}4 & 5 & 7 & 2\\1 & 2 & -2 & -1\end{bmatrix} \to \begin{bmatrix}4 & 5 & 7 & 2\\0 & -3 & 15 & 6\end{bmatrix}$$

$$\to \begin{bmatrix}4 & 5 & 7 & 2\\0 & 1 & -5 & 2\end{bmatrix} \to \begin{bmatrix}4 & 0 & 32 & 12\\0 & 1 & -5 & -2\end{bmatrix} \to \begin{bmatrix}1 & 0 & 8 & 3\\0 & 1 & -5 & -2\end{bmatrix}$$

(first I added -4 times the second row to the first, then I divided the second row by -3, then I subtracted 5 times the second row from the first, and finally I divided the first row by 4)

Hence:

$$\mathcal{C} \overset{P}{\leftarrow} \mathcal{B} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}$$

We have:

$$[\mathbf{x}]_{\mathcal{C}} = \mathcal{C} \overset{P}{\leftarrow} \mathcal{B} \, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 20 \\ -13 \end{bmatrix}$$

5. *(10 points)* Define $T : M_{2\times 2} \to M_{2\times 2}$ by:

$$T(A) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} A$$

Find the matrix of T relative to the basis:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ of } M_{2\times 2}$$

$$T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

Hence the matrix of T is:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Note: If you wrote $\begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$, I took off 4 points! It's **very** important to convert the vectors you found into *column* vectors!!!

6. *(10 points)* Define $T : C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R})$ (the space of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$) by:

$$T(y) = y'' - 5y' + 6y$$

(a) *(5 points)* Show that T is a linear transformation

$$T(y_1+y_2) = (y_1+y_2)''-5(y_1+y_2)'+6(y_1+y_2) = y_1''-5y_1'+6y_1+y_2''-5y_2'+6y_2 = T(y_1)+T(y_2)$$

$$T(cy) = (cy)'' - 5(cy)' + 6(cy) = c(y'' - 5y' + 6y) = cT(y)$$

(b) *(5 points)* Find a basis for $Ker(T)$ (or $Nul(T)$ if you wish). Show that the basis you found is in fact a basis!

Remember $Ker(T) = \{y \mid T(y) = 0\} = \{y \mid y'' - 5y' + 6y = 0\}$

Now let's solve $y'' - 5y' + 6y = 0$. The auxiliary equation is $r^2 - 5r + 6 = (r-2)(r-3) = 0$, which gives $r = 2, r = 3$, hence the general solution is:

$$y(t) = Ae^{2t} + Be^{3t}$$

We claim that $\{e^{2t}, e^{3t}\}$ is a basis for $Ker(T)$. We already showed it spans $Ker(T)$, so all we need to show that it is linearly independent.

But the Wronskian matrix of e^{2t}, e^{3t} is $\tilde{W}(t) = \begin{bmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{bmatrix}$, hence $\tilde{W}(0) = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, so $W(0) = 3 - 2 = 1 \neq 0$, hence e^{2t} and e^{3t} are linearly independent!

Therefore $\{e^{2t}, e^{3t}\}$ is linearly independent and Spans $Ker(T)$, and hence it's a basis for $Ker(T)$.

7. *(5 points)* Find the largest *open* interval (a, b) on which the following differential equation has a unique solution:

$$(x-3)y'' + \left(\sqrt{x}\right) y' = \sqrt{x-1}$$

with

$$y(2) = 3, y'(2) = 1$$

First convert the equation in standard form:

$$y'' + \left(\frac{\sqrt{x}}{x-3}\right) y' = \frac{\sqrt{x-1}}{x-3}$$

Now let's look at the domain of each term:

The domain of $\frac{\sqrt{x}}{x-3}$ is $[0, 3) \cup (3, \infty)$ (i.e. all nonnegative real numbers except 3). The part of that interval which contains the initial condition 2 is $\boxed{[0, 3)}$

The domain of $\frac{\sqrt{x-1}}{x-3}$ is $[1, 3) \cup (3, \infty)$. The part of that which contains the initial condition 2 is $\boxed{[1, 3)}$

And if you intersect the two domains you found you get $[1, 3)$, and hence the answer is $\boxed{(1, 3)}$ (remember we want an open interval!).

8. *(10 points)* Solve the following differential equation:

$$y''' - 3y'' + 12y' - 10y = 0$$

The auxiliary equation is $r^3 - 3r^2 + 12r - 10 = 0$.

Now, by the rational roots theorem, we know that if the above polynomial has a rational root, then $r = \frac{a}{b}$, where a divides the constant term -42 and b divides the leading term 1.

The only integers which divide -10 are $\pm 1, \pm 2, \pm 5, \pm 10$
And the only integers which divide 1 are ± 1. Hence our guesses are: $\pm 1, \pm 2, \pm 5, \pm 10$.

If you plug-and-chug, you eventuall figure out that $r = 1$ works (the first guess!), i.e. $r = 1$ is a root of the auxiliary polynomial.

Now all you have to do is use long division and divide $r^3 - 3r^2 + 12r - 10$ by $r - 1$

$$\begin{array}{r|l} & X^2 - 2X + 10 \\ \hline X - 1 & X^3 - 3X^2 + 12X - 10 \\ & -X^3 + X^2 \\ \hline & -2X^2 + 12X \\ & 2X^2 - 2X \\ \hline & 10X - 10 \\ & -10X + 10 \\ \hline & 0 \end{array}$$

In other words, $r^3 - 3r^2 + 12r - 10 = (r - 1)(r^2 - 2r + 10)$

Now $r^2 - 2r + 10 = 0 \Rightarrow r = \frac{2 \pm \sqrt{-36}}{2} = 1 \pm 3i$.

Hence, the roots of $r^3 - 3r^2 + 12r - 10$ are $r = 1$ and $r = 1 \pm 3i$. Hence the general solution of the differential equation is:

$$y(t) = Ae^t + Be^t \cos(3t) + Ce^t \sin(3t)$$

9. *(8 points)* Solve the following differential equation:

$$y'' - 3y' + 2y = e^{3t}$$

(a) *(4 points)* Using undetermined coefficients

First let's solve the homogeneous equation: $r^2 - 3r + 2 = (r-1)(r-2) = 0 \Rightarrow r = 1, r = 2$, hence $y_0(t) = Ae^t + Be^{2t}$

Now for the particular solution y_p, let's guess $y_p(t) = Ce^{3t}$. If you plug this into the original equation, you get:

$$9Ce^{3t} - 9Ce^{3t} + 2Ce^{3t} = e^{3t} \Rightarrow 2C = 1 \Rightarrow C = \frac{1}{2}$$

Hence $y_p(t) = \frac{1}{2}e^{3t}$, and $\boxed{y(t) = Ae^t + Be^{2t} + \frac{1}{2}e^{3t}}$

(b) *(4 points)* Using variation of parameters

We already found $y_0(t) = Ae^t + Be^{2t}$, now suppose $y_p(t) = v_1(t)e^t + v_2(t)e^{2t}$, then:

$$\begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}$$

Hence:

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix} = \begin{bmatrix} 2e^{-t} & -e^{-t} \\ -e^{-2t} & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix} = \begin{bmatrix} -e^{2t} \\ e^t \end{bmatrix}$$

Hence $v_1'(t) = -e^{2t}$, so $v_1(t) = -\frac{1}{2}e^{2t}$, and $v_2'(t) = e^t$, so $v_2(t) = e^t$.

And thus $y_p(t) = -\frac{1}{2}e^{2t}e^t + e^te^{2t} = -\frac{1}{2}e^{3t} + e^{3t} = \frac{1}{2}e^{3t}$

And hence: $\boxed{y(t) = Ae^t + Be^{2t} + \frac{1}{2}e^{3t}}$

10. *(5 points)*

(a) *(1 point)* If $T : V \to W$ is a one-to-one linear transformation and $T(\mathbf{x}) = \mathbf{0}$, what can you say about $\mathbf{x}$?

$\mathbf{x} = \mathbf{0}$

(b) *(4 points)* Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly *independent* vectors (in V) and $T : V \to W$ is a *one-to-one* linear transformation. Show that $T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})$ are also linearly independent.

Suppose $aT(\mathbf{u}) + bT(\mathbf{v}) + cT(\mathbf{w}) = \mathbf{0}$.

We want to show that $a = b = c = 0$

Then $T(a\mathbf{u} + b\mathbf{v} + c\mathbf{w}) = \mathbf{0}$ since T is linear

Hence $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$ since T is one-to-one (this is (a), with $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$

Hence $a = b = c = 0$ since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent!

11. *(2 points)* What is the basis of your happiness? :)

Anything from ‘chocolate’ to ‘Math 54’ gets full credit :) However, if you didn’t write anything, you get 0 points :(

Bonus: In this problem, we’re going to use the Wronskian to find the general solution of a quite complicated differential equation! This should illustrate yet again the power of the Wronskian!

(a) Consider the differential equation:

$$y'' + P(t)y' + Q(t)y = 0$$

Recall the definition of the Wronskian determinant:

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_2'(t)y_1(t) - y_1'(t)y_2(t)$$

Where y_1 and y_2 solve the above differential equation.

By differentiating $W(t)$ with respect to t, find a simple differential equation satisfied by $W(t)$ and solve it. You answer will involve the $\int$ sign!

$$\begin{aligned} W'(t) =& y_2''(t)y_1(t) + \cancel{y_2'(t)y_1'(t)} - y_1''(t)y_2(t) - \cancel{y_1'(t)y_2'(t)} \\ =& y_2''(t)y_1(t) - y_1''(t)y_2(t) \\ =& \left(-P(t)y_2'(t) - Q(t)y_2(t)\right)y_1(t) + \left(P(t)y_1'(t) + Q(t)y_1(t)\right)y_2(t) \\ =& -P(t)y_2'(t)y_1(t) - \cancel{Q(t)y_1(t)y_2(t)} + P(t)y_1'(t)y_2(t) + \cancel{Q(t)y_1(t)y_2(t)} \\ =& -P(t)\left(y_2'(t)y_1(t) - y_1'(t)y_2(t)\right) \\ =& -P(t)W(t) \end{aligned}$$

Hence $W'(t) = -P(t)W(t)$, so $\boxed{W(t) = Ce^{-\int P(t)dt}}$

(b) Hence, by (a), we know:

$$y_2'(t)y_1(t) - y_2(t)y_1'(t) = \text{your answer in } (a)$$

Divide this equality by $(y_1(t))^2$ and recognize the left-hand-side as the derivative of a quotient, and hence solve for y_2 in terms of y_1. You answer will involve another $\int$ sign!

We have:

$$
\begin{aligned}
y_2'(t)y_1(t) - y_2(t)y_1'(t) =& e^{-\int P(t)dt} \\
\frac{y_2'(t)y_1(t) - y_2(t)y_1'(t)}{(y_1(t))^2} =& \frac{e^{-\int P(t)dt}}{(y_1(t))^2} \\
\left(\frac{y_2(t)}{y_1(t)}\right)' =& \frac{e^{-\int P(t)dt}}{(y_1(t))^2} \\
\frac{y_2(t)}{y_1(t)} =& \int \frac{e^{-\int P(t)dt}}{(y_1(t))^2} \\
y_2(t) =& \left(\int \frac{e^{-\int P(t)dt}}{(y_1(t))^2}\right) y_1(t)
\end{aligned}
$$

(c) Let's apply the result in (b) to the differential equation:

$$y'' - \tan(t)y' + 2y = 0$$

(here $P(t) = -\tan(t)$, $Q(t) = 2$)

One solution (by guessing) is given by $y_1(t) = \sin(t)$. Use your answer to (b) to find *another* solution $y_2(t)$!

Hint: You may use the following facts: $\int \tan(t)dt = -\ln(\cos(t))$, the substitution $u = \frac{1}{\sin(t)}$, and finally the formula $\frac{u^2}{1-u^2} = \frac{1}{1-u^2} - 1 = \frac{1}{2(1-u)} + \frac{1}{2(1+u)} - 1$.

We have:

$$
\begin{aligned}
y_2(t) &= \left(\int \frac{e^{-\int P(t)dt}}{(y_1(t))^2}\right) y_1(t) \\
&= \left(\int \frac{e^{\int \tan(t)dt}}{\sin^2(t)}\right) \sin(t) \\
&= \left(\int \frac{e^{-\ln(\cos(t))}}{\sin^2(t)}\right) \sin(t) \\
&= \left(\int \frac{\frac{1}{e^{\ln(\cos(t))}}}{\sin^2(t)}\right) \sin(t) \\
&= \left(\int \frac{\frac{1}{\cos(t)}}{\sin^2(t)}\right) \sin(t) \\
&= \left(\int \frac{1}{\cos(t)\sin^2(t)}\right) \sin(t) \\
&= \left(\int \left(\frac{1}{\cos^2(t)}\right)\left(\frac{\cos(t)}{\sin^2(t)}\right)\right) \sin(t) \\
&= \left(\int \left(\frac{1}{1-\sin^2(t)}\right)\left(\frac{\cos(t)}{\sin^2(t)}\right)\right) \sin(t) \\
&= \left(\int \frac{-du}{1-\frac{1}{u^2}}\right) \sin(t) \qquad \text{Use } u = \frac{1}{\sin(x)}, \text{ then } \sin(x) = \frac{1}{u} \\
&= \left(\int \frac{-u^2}{u^2-1} du\right) \sin(t) \qquad \text{(multiply top and bottom by } u^2\text{)} \\
&= \left(\int -1 + \frac{-1}{2(u-1)} + \frac{1}{2(1+u)}\right) \sin(t) \\
&= \left(-u + \frac{1}{2}\ln|u+1| - \frac{1}{2}\ln|u-1|\right) \sin(t) \\
&= \left(\frac{-1}{\sin(t)} + \frac{1}{2}\ln\left|\frac{1}{\sin(t)}+1\right| - \frac{1}{2}\ln\left|\frac{1}{\sin(t)}-1\right|\right) \sin(t) \\
&= \left(\frac{-1}{\sin(t)} + \frac{1}{2}\ln\left|\frac{\frac{1}{\sin(t)}+1}{\frac{1}{\sin(t)}-1}\right|\right) \sin(t) \\
&= \left(\frac{1}{-\sin(t)} + \frac{1}{2}\ln\left|\frac{1+\sin(t)}{1-\sin(t)}\right|\right) \sin(t) \qquad \text{(multiply top and bottom by } \sin(t)\text{)} \\
&= -1 + \sin(t)\coth^{-1}(\sin(t))
\end{aligned}
$$

(d) Notice that the equation $y'' - \tan(t)y' + 2y = 0$, although quite complicated, is still *linear*. What is the general solution of $y'' - \tan(t)y' + 2y = 0$?

$$y(t) = Ay_1(t) + By_2(t) = A\sin(t) + B(-1 + \sin(t)\coth^{-1}(\sin(t)))$$